STRESS – STRAIN STATE OF A PARABOLIC SHELL OF REVOLUTION SUBJECTED TO EXTERNAL MAGNETIC PRESSURE

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The paper investigates the stress–strain state of an axisymmetrically loaded shell which arises when a strong electric current flows in it. The shell with current is an element of a system intended for focusing π and K mesons in neutron experiments. The problem is solved by numerical integration on a computer of equations of the theory of shells by the two-sided matrix run-through method, and also by an approximate analytical solution. The algorithm being applied can be used to calculate an arbitrary shell of revolution of variable thickness. The results thus obtained are discussed.

In investigations of high-energy physics, systems of current-carrying shells are used to form particle beams. Such systems usually constitute a collection of shells of revolution subjected to impulsive electric current of hundreds of kiloamperes.

In [1] a system consisting of current parabolic lenses [2] is proposed for the focusing of π and K mesons in a neutron experiment. For the parameters of this system the pressure of the magnetic field of the current amounts to several hundreds of atmospheres, and in the case of a small thickness of the shell the mechanical stresses can be very high.

A parabolic lens in the general case is a body of revolution with a variable wall thickness (Fig. 1) loaded by an axisymmetric magnetic pressure which is varying with time and is inhomogeneous in space. The lens consists of two paraboloids of revolution 1, joined at the vertices via a constructional neck 2, flanges 3, and a coaxial current carrier 4 with current I_0. In the case being considered the thickness of its walls h varying over the lens is considerably less than the characteristic radius of curvature R of the surface, while the frequency of natural vibrations of the lens exceeds the frequency of current variation. In view of this, for the description of the stress–strain state we will proceed from the system of equations of the theory of shells of revolution for a static nature of the load (see, for example, [3]).

An exact analytical solution of the system is known only for certain particular types of shells [4]. To solve the problem, two methods are used below: a numerical solution of the complete system of equations of the shell theory, and an approximate analytical solution.

1. The system of equations for a shell of revolution loaded by an axisymmetric pressure has the form [3]

$$
\begin{gathered}
\frac{1}{R_1}\frac{dT_1}{d\theta}+(T_1-T_2)\frac{\operatorname{ctg}\theta}{R_2}-\frac{Q_1}{R_1}=0\\
\frac{T_1}{R_1}+\frac{T_2}{R_2}+\frac{1}{R_1}\frac{dQ_1}{d\theta}+Q_1\frac{\operatorname{ctg}\theta}{R_2}=p_n\\
\frac{1}{R_1}\frac{dM_1}{d\theta}+(M_1-M_2)\frac{\operatorname{ctg}\theta}{R_2}=Q_1\\
T_1=\frac{Eh}{1-\mu^2}\left[\frac{1}{R_1}\left(\frac{du}{d\theta}+w\right)+\frac{\mu}{R_2}(u\operatorname{ctg}\theta+w)\right]\\
T_2=\frac{Eh}{1-\mu^2}\left[\frac{1}{R_2}(u\operatorname{ctg}\theta+w)+\frac{\mu}{R_1}\left(\frac{du}{d\theta}+w\right)\right]
\end{gathered}
\tag{1.1}
$$

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$$M_1 = \frac{Eh^3}{12(1-\mu^2)}\left[\frac{1}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dw}{d\theta} - \frac{u}{R_1}\right) + \frac{\mu \operatorname{ctg}\theta}{R_1R_2}\left(\frac{dw}{d\theta} - u\right)\right]$$
$$M_2 = \frac{Eh^3}{12(1-\mu^2)}\left[\frac{\operatorname{ctg}\theta}{R_1R_2}\left(\frac{dw}{d\theta} - u\right) + \frac{\mu}{R_1}\frac{d}{d\theta}\left(\frac{1}{R_1}\frac{dw}{d\theta} - \frac{u}{R_1}\right)\right]$$

where E is modulus of elasticity; μ is Poisson's ratio; T_1, T_2, M_1, M_2 are forces and moments per unit length,

$$T_1 = \int_{-h/2}^{h/2} \sigma_1 \, d\xi, \quad T_2 = \int_{-h/2}^{h/2} \sigma_2 \, d\xi, \quad M_1 = \int_{-h/2}^{h/2} \sigma_1 \xi \, d\xi, \quad M_2 = \int_{-h/2}^{h/2} \sigma_2 \xi \, d\xi$$

σ_1 and σ_2 are the meridional and circumferential stresses; ξ are the coordinates across the thickness of the shells from its middle surface; ($\xi = h/2$ corresponds to the inner surface of the shell, $\xi = -h/2$—ee corresponds to its outer surface); u and w are the displacements of points of the middle surface in the meridian and normal directions; Q_1 is the shear force per unit length; R_1 and R_2 are the radii of curvature of the middle surface; θ is the angle coordinate along the meridian. The positive directions of the forces, moments, and displacements are shown on the element of the shell (Fig. 2); p_n is the pressure of the magnetic field, normal to the outer surface of the shell; it is given by the expression

$$p_n = 10^4(1/2\pi)I_0^2 r^{-2} \tag{1.2}$$

p_n in kgf/cm^2, I_0 is the amplitude value of the current in mA, r is the current radius of the parallel of the outer surface of the shell in centimeters.

The problem can be reduced to the solution of six ordinary differential equations with variable coefficients in the case of two-point boundary conditions — with three on each edge of the paraboloidal portion of the lens. For the numerical solution of the boundary-value problem whose homogeneous differential equations have, along with decreasing solutions, rapidly increasing solutions (as a rule, problems of the shell theory belong to them), we use the run-through method. One of the possible and most effective variants of it — the two-sided matrix run-through method [5, 6] — was used to solve the given problem.

The essence of the method consists of the following. The total vector **X** of 2n sought functions, characterizing the state of an arbitrary section of the system, is divided into two vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ with n functions in each. At each point s of the interval of integration (s_0, s_k), between the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ we can find two linear relations

$$\mathbf{X}_1(s) = \mathbf{L}_i(s)\mathbf{X}_2(s) + \mathbf{R}_i(s) \quad (i = 1, 2) \tag{1.3}$$

where $\mathbf{L}_i(s)$ are square (n× n) matrices, and $\mathbf{R}_i(s)$ are vectors of n coefficients. $\mathbf{L}_i(s)$ and $\mathbf{R}_i(s)$ are determined by integration from s_0 to s_k ($ds > 0$, $i = 1$) and from s_k to s_0 ($ds < 0$, $i = 2$) of the corresponding differential equations, obtained, for example, in [7] for a variant of one-sided run-through method, with the initial conditions following from the fixing conditions of the edges. By this the left and right boundary conditions are transferred to all points s of the interval of integration, with the geometry of the portions, respectively, on the left and right of s and the external load applied to these sections taken into account. In this manner the direct run-through ($i = 1$) and reverse run-through ($i = 2$) are accomplished.

The physical meaning of $\mathbf{L}_i$ and $\mathbf{R}_i$ depends on the choice of the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$. Thus, for $i = 1$, if $\mathbf{X}_1$ is a displacement vector and $\mathbf{X}_2$ is the vector of internal force quantities, then $\mathbf{L}_1$ is a flexibility matrix of the part of the system from s_0 to s, while $\mathbf{R}_1$ are displacements of the sections caused by the load applied on the left from it. Conversely, if $\mathbf{X}_1$ is a vector of force quantities and $\mathbf{X}_2$ is a displacement vector, then $\mathbf{L}_1$ is a stiffness matrix, while $\mathbf{R}_1$ are the force quantities in the section s caused by the load applied on the left of s. In the case of a mixed content of the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ different components of the matrix $\mathbf{L}_1$ and the vector $\mathbf{R}_1$ will have a different physical meaning.

The choice of the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ is determined by the fixing conditions of the boundary $s = s_0$ from which we start the solution of the problem; in particular, the n functions specified on this boundary are included in the $\mathbf{X}_1$ vector. During the reverse run-through the content of the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ remains the same, since after the forward run-through to the boundary $s = s_k$ all 2n sought functions are determined. For the reverse run-through we can formally use in the role of initial condition those n conditions of them which for $s = s_k$ completely define the vector $\mathbf{X}_1$. A simultaneous solution of Eqs. (1.3) obtained during the direct run-through ($i = 1$) and the reverse run-through ($i = 2$) gives the sought values of $\mathbf{X}_1(s)$ and $\mathbf{X}_2(s)$ at any point s of the interval (s_0, s_k).

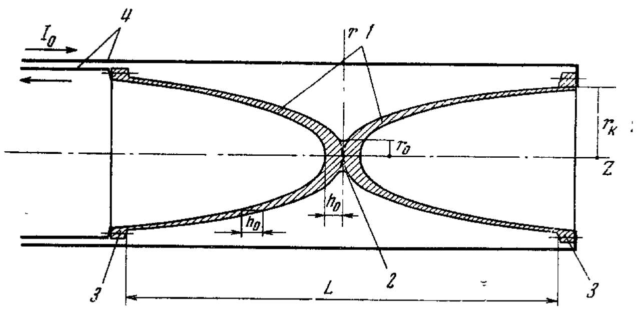

Fig. 1

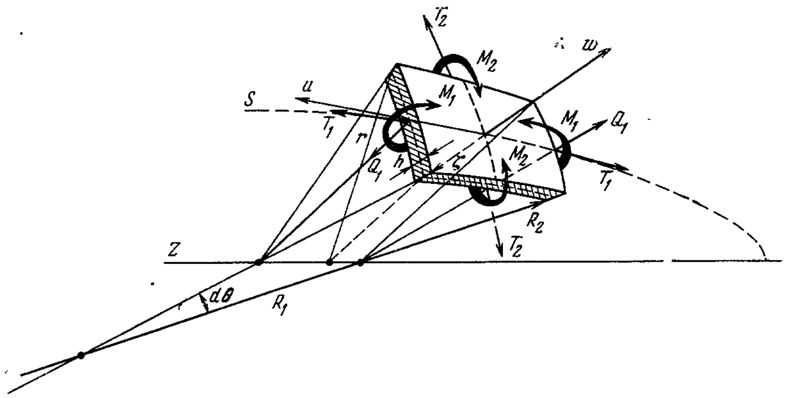

Fig. 2

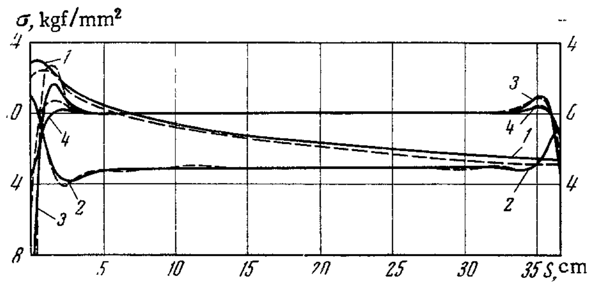

Fig. 3

We next consider in the general form an application of the given method for the calculation of composite constructions; this is necessary for the calculation of joined shells of revolution. We assign, in the zone joining the two parts, the index 1 to the left part and the index 2 to the right part of them (upper indices). Then the joining conditions of the two parts can be represented in the form

$$\mathbf{M}^{+}\mathbf{X}^{(1)} = \mathbf{X}^{(2)} \tag{1.4}$$

$$\mathbf{M}^{-}\mathbf{X}^{(2)} = \mathbf{X}^{(1)} \tag{1.5}$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are the complete (2n) vectors of state of the joining section; $\mathbf{M}^{+}$ and $\mathbf{M}^{-}$ are square ($2n\times 2n$) transformation matrices. The expressions (1.4) and (1.5) constitute the equations of equilibrium of an element of the joint and the continuity of strains. It is obvious that

$$\mathbf{M}^{+}\mathbf{M}^{-} = \mathbf{M}^{-}\mathbf{M}^{+} = \mathbf{I} \tag{1.6}$$

where I is a unit ($2n\times 2n$) matrix.

Carrying out a forward run-through, we have at the joint, for the part 1, the relation

$$\mathbf{X}_1^{(1)} = \mathbf{L}_1^{(1)}\mathbf{X}_2^{(1)} + \mathbf{R}_1^{(1)} \tag{1.7}$$

which must be transformed by means of (1.4) or (1.5) into

$$\mathbf{X}_1^{(2)} = \mathbf{L}_1^{(2)} \mathbf{X}_2^{(2)} + \mathbf{R}_1^{(2)} \tag{1.8}$$

for the part 2. The transformation (1.4) is written in the form

$$\mathbf{M}_{11}^{+}\mathbf{X}_1^{(1)} + \mathbf{M}_{12}^{+}\mathbf{X}_2^{(1)} = \mathbf{X}_1^{(2)}, \quad \mathbf{M}_{21}^{+}\mathbf{X}_1^{(1)} + \mathbf{M}_{22}^{+}\mathbf{X}_2^{(1)} = \mathbf{X}_2^{(2)} \tag{1.9}$$

where $\mathbf{M}_{ik}^{+}$ are square $(n\times n)$ blocks in the matrix $\mathbf{M}^{+}$. Substituting (1.7) into (1.9), we obtain

$$\mathbf{X}_1^{(2)} = [\mathbf{M}_{11}^{+}\mathbf{L}_1^{(1)} + \mathbf{M}_{12}^{+}]\,[\mathbf{M}_{21}^{+}\mathbf{L}_1^{(1)} + \mathbf{M}_{22}^{+}]^{-1}\mathbf{X}_2^{(2)} + \mathbf{M}_{11}^{+}\mathbf{R}_1^{(1)} - [\mathbf{M}_{11}^{+}\mathbf{L}_1^{(1)} + \mathbf{M}_{12}^{+}]\,[\mathbf{M}_{21}^{+}\mathbf{L}_1^{(1)} + \mathbf{M}_{22}^{+}]^{-1}\,\mathbf{M}_{21}^{+}\mathbf{R}_1^{(1)} \tag{1.10}$$

A comparison of (1.10) with (1.9) gives

$$\begin{aligned} \mathbf{L}_1^{(2)} &= [\mathbf{M}_{11}^{+}\mathbf{L}_1^{(1)} + \mathbf{M}_{12}^{+}]\,[\mathbf{M}_{21}^{+}\mathbf{L}_1^{(1)} + \mathbf{M}_{22}^{+}]^{-1} \\ \mathbf{R}_1^{(2)} &= [\mathbf{M}_{11}^{+} - \mathbf{L}_1^{(2)}\mathbf{M}_{21}^{+}]\,\mathbf{R}_1^{(1)} \end{aligned} \tag{1.11}$$

If we use the joining conditions in the form (1.5), then for $\mathbf{L}_1^{(2)}$ and $\mathbf{R}_1^{(2)}$ we have

$$\begin{aligned} \mathbf{L}_1^{(2)} &= -\,[\mathbf{M}_{11}^{-} - \mathbf{L}_1^{(1)}\mathbf{M}_{21}^{-}]^{-1}\,[\mathbf{M}_{12}^{-} - \mathbf{L}_1^{(1)}\mathbf{M}_{22}^{-}] \\ \mathbf{R}_1^{(2)} &= [\mathbf{M}_{11}^{-} - \mathbf{L}_1^{(1)}\mathbf{M}_{21}^{-}]^{-1}\,\mathbf{R}_1^{(1)} \end{aligned} \tag{1.12}$$

where $\mathbf{M}_{ik}^{-}$ are square $(n\times n)$ blocks in the matrix $\mathbf{M}^{-}$. The expressions for $\mathbf{L}_2^{(1)}$ and $\mathbf{R}_2^{(1)}$ for transition from the part 2 to the part 1 during the reverse run-through can be obtained analogously, or from expressions (1.11) and (1.12), replacing the index of forward run-through 1 of $\mathbf{L}_i$ and $\mathbf{R}_i$ by the index of reverse run-through 2 (lower indices). We obtain

$$\begin{aligned} \mathbf{L}_2^{(1)} &= -\,[\mathbf{M}_{11}^{+} - \mathbf{L}_2^{(2)}\mathbf{M}_{21}^{+}]^{-1}\,[\mathbf{M}_{12}^{+} - \mathbf{L}_2^{(2)}\mathbf{M}_{22}^{+}] \\ \mathbf{R}_2^{(1)} &= [\mathbf{M}_{11}^{+} - \mathbf{L}_2^{(2)}\mathbf{M}_{21}^{+}]^{-1}\,\mathbf{R}_2^{(2)} \end{aligned} \tag{1.13}$$

$$\begin{aligned} \mathbf{L}_2^{(1)} &= [\mathbf{M}_{11}^{-}\mathbf{L}_2^{(2)} + \mathbf{M}_{12}^{-}]\,[\mathbf{M}_{21}^{-}\mathbf{L}_2^{(2)} + \mathbf{M}_{22}^{-}]^{-1} \\ \mathbf{R}_2^{(1)} &= [\mathbf{M}_{11}^{-} - \mathbf{L}_2^{(1)}\mathbf{M}_{21}^{-}]\,\mathbf{R}_2^{(2)} \end{aligned} \tag{1.14}$$

From expressions (1.11)-(1.14) we see that for $\mathbf{L}_1$ and $\mathbf{R}_1$ during the forward and reverse run-through we can use either expressions of identical form, whose blocks $\mathbf{M}_{ik}$ of the same term belong to different, mutually inverse matrices $\mathbf{M}^{+}$ and $\mathbf{M}^{-}$, or expressions of different form, but with the use of blocks of only one of the matrices $\mathbf{M}^{+}$ or $\mathbf{M}^{-}$ for both passages. By means of relations (1.11)-(1.14), by the method of two-sided run-through, we can calculate constructions formed from an artibtrary numer of joined parts.

Thus, in the method of two-sided matrix run-through the solution of the original boundary-value problem is reduced to the solution of four Cauchy problems for $\mathbf{L}_1(s)$, $\mathbf{R}_1(s)$ and $\mathbf{L}_2(s)$, $\mathbf{R}_2(s)$, which are solvable in pairs on the left and on the right of the interval (s_0, s_k) with the corresponding initial conditions. No rapidly increasing functions will arise in any of the problems; this can be shown in each concrete case.

The advantage of the method of two-sided run-through in comparison with the method of usual run-through [7] consists of the fact that the differential equations for $\mathbf{L}_i$ and $\mathbf{R}_i$ $(i=1, 2)$ are integrated on the left and on the right of the interval (s_0, s_k) independently; therefore, the coefficient matrices $\mathbf{L}_i$ and $\mathbf{R}_i$ are not stored on each step of integration, but only at the points for which we have to obtain the solution $\mathbf{X}_1$, $\mathbf{X}_2$. As a result, the volume of the memory of the computer being used is substantially reduced.

2. To realize the method just described, programs were set up which were suitable for the calculations of an arbitrary shell of revolution of variable thickness, and shells joined together. The programs were written in FORTRAN for a BÉSM-6 computer. The differential matrix equations for $\mathbf{L}_i$ and $\mathbf{R}_i$ were integrated by the Runge–Kutta method. The initial system of equations (1.1), transformed into a fundamental system of six differential equations of the first order (axisymmetric problem, $n=3$) in a mixed form, when brought to a dimensionless form is written as

$$\begin{aligned} \frac{du^\circ}{ds^\circ} &= T_1^\circ \frac{B_0}{B} - u^\circ\mu \frac{R_0}{R_2}\operatorname{ctg}\theta - w^\circ\left(\frac{R_0}{R_1} + \mu\frac{R_0}{R_2}\right) \\ \frac{dw^\circ}{ds^\circ} &= \vartheta_1 + u^\circ\frac{R_0}{R_1}, \quad \frac{d\vartheta_1}{ds^\circ} = M_1^\circ\frac{D_0}{D} - \vartheta_1\mu\frac{R_0}{R_2}\operatorname{ctg}\theta \end{aligned}$$

$$\frac{dT_1^\circ}{ds^\circ} = -T_1^\circ(1-\mu)\frac{R_0}{R_2}\operatorname{ctg}\theta + Q_1^\circ\frac{R_0}{R_2} + u^\circ(1-\mu^2)\frac{B}{B_0}\left(\frac{R_0}{R_2}\right)^2\operatorname{ctg}^2\theta + w^\circ(1-\mu^2)\frac{B}{B_0}\left(\frac{R_0}{R_2}\right)^2\operatorname{ctg}\theta - q_\theta^\circ$$

$$\frac{dQ_1^\circ}{ds^\circ} = -Q_1^\circ\frac{R_0}{R_2}\operatorname{ctg}\theta - T_1^\circ\left(\frac{R_0}{R_1} + \mu\frac{R_0}{R_2}\right) - u^\circ(1-\mu^2)\frac{B}{B_0}\left(\frac{R_0}{R_2}\right)^2\operatorname{ctg}\theta - w^\circ(1-\mu^2)\frac{B}{B_0}\left(\frac{R_0}{R_2}\right)^2 + q_n^\circ$$

$$\frac{dM_1^\circ}{ds^\circ} = -M_1^\circ(1-\mu)\frac{R_0}{R_2}\operatorname{ctg}\theta + Q_1^\circ\frac{1}{c_0^2} + \vartheta_1(1-\mu^2)\frac{D}{D_0}\left(\frac{R_0}{R_2}\right)^2\operatorname{ctg}^2\theta \tag{2.1}$$

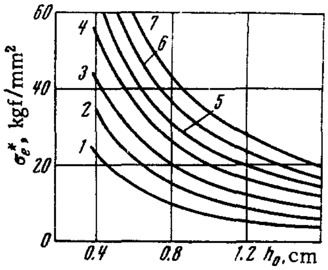

Fig. 4

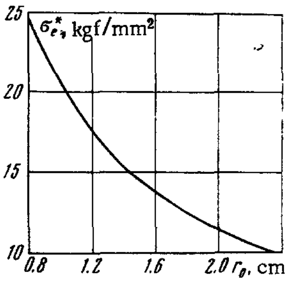

Fig. 5

or in the matrix form

$$dX / ds^{\circ} = \mathbf{F}\mathbf{X} + \mathbf{G} \tag{2.2}$$

In (2.2) $\mathbf{X}$ is a complete (2n=6) vector of sought functions, with

$$\mathbf{X}_1 = \begin{bmatrix} u^{\circ} \\ w^{\circ} \\ \vartheta_1 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} T_1{}^{\circ} \\ Q_1{}^{\circ} \\ M_1{}^{\circ} \end{bmatrix}$$

where $\mathbf{F}$ is the matrix of variable coefficients, and $\mathbf{G}$ is the vector of load functions. In the case being considered $q_{\theta}{}^{\circ} = 0$, $q_n{}^{\circ} = p_n{}^{\circ}$.

The peculiarity of writing the system in the form (2.1) is the absence, in explicit form, of differentiation of the geometrical and stiffness parameters of the shell, the direct way in which the sought quantities are obtained as a result of the solution, and also the simplicity in specifying the boundary and joining conditions. In system (2.1) the length of the arc of the meridian s is taken as the independent variable. The rest of the sought quantities (T_2, M_2) are calculated, after solving the basic system (2.1), from additional algebraic relationships. They have the form

$$T_2{}^{\circ} = \mu T_1{}^{\circ} + (1 - \mu^2) \frac{B}{B_0} \frac{R_0}{R_2} (u^{\circ} \operatorname{ctg} \theta + w^{\circ})$$
$$M_2{}^{\circ} = \mu M_1{}^{\circ} + (1 - \mu^2) \frac{D}{D_0} \frac{R_0}{R_2} \vartheta_1 \operatorname{ctg} \theta \tag{2.3}$$

In (2.1)-(2.3) we have used the notation

$$u^{\circ} = \frac{u}{R_0}, \quad w^{\circ} = \frac{w}{R_0}, \quad T_1{}^{\circ} = \frac{T_1}{B_0}, \quad T_2{}^{\circ} = \frac{T_2}{B_0}$$
$$Q_1{}^{\circ} = \frac{Q_1}{B_0}, \quad M_1{}^{\circ} = \frac{M_1 R_0}{D_0}, \quad M_2{}^{\circ} = \frac{M_2 R_0}{D_0}, \quad c_0{}^2 = \frac{D_0}{B_0 R_0{}^2}$$
$$s^{\circ} = \frac{s}{R_0}, \quad p_n{}^{\circ} = \frac{p_n R_0}{B_0}, \quad B_0 = \frac{E_0 h_0}{1 - \mu^2}, \quad B = \frac{Eh}{1 - \mu^2}$$
$$D_0 = \frac{E_0 h_0{}^3}{12(1 - \mu^2)}, \quad D = \frac{Eh^3}{12(1 - \mu^2)}$$

where ϑ_1 is the angle of rotation of the normal to the middle surface of the shell in the direction of the meridian; h_0, h, B_0, B, D_0, D are the thicknesses, tensile, and flexural stiffnesses at the reference point 0 and the current point s; E_0, E are the corresponding moduli of elasticity, where in the general case $E(s) \neq$ const is possible; R_0 is the reference radius.

For a parabolic lens (Fig. 1) with a constant of parabola a, it is convenient to take the point at $\theta = 0$ as the reference point. Here $R_0 = 1/2a$ and then $R_0/R_1 = \cos^3\theta$, $R_0/R_2 = \cos\theta$, $B/B_0 = \cos\theta$, $D/D_0 = \cos^3\theta$, [for E(s) = const]. In the case where two or more shells are joined, we can choose a common reference point for all of them. In the problem being solved here, where a parabolic shell joins the neck of the lens in the form of a cylindrical shell, we have taken the point of the paraboloid at $\theta = 0$ in the role of such a point.

The blocks of the matrices $\mathbf{M}^+$ and $\mathbf{M}^-$, for the joint between the cylindrical and parabolic shells, in the case of the chosen vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ have the form

$$\mathbf{M}_{11}{}^{+} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_{22}{}^{+} = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{M}_{11}{}^{-} = \mathbf{M}_{22}{}^{+}$, $\mathbf{M}_{22}{}^{-} = \mathbf{M}_{11}{}^{+}$. The remaining blocks are zero. The solution of the problem here was started from the right boundary (s_k, θ_k, r_k). In the role of the left boundary conditions we took the conditions at the middle of the cylindrical shell, which follow from the symmetry of the construction. Here the displacements along the u° axis, the angle of rotation of the normal to the middle surface along the meridian ϑ_1, and the shear force $Q_1{}^{\circ}$ will be zero.

After the solution of the system (2.1) and the computation of T_2 and M_2 according to (2.3), the maximum stresses in the section are determined from the relation [3]:

$$\sigma_1 = \frac{T_1^\circ B_0}{h} \pm \frac{6M_1^\circ D_0}{R_0 h^2}, \quad \sigma_2 = \frac{T_2^\circ B_0}{h} \pm \frac{6M_2^\circ D_0}{R_0 h^2} \tag{2.4}$$

in which the minus sign corresponds to compressive stresses, and the plus corresponds to tension. Since there is a complex stress state in the shells, for its numerical estimate we go to a uniaxial equivalent stress state, in accordance with the theory of maximum shear stresses, according to the expression [6]:

$$\sigma_e = {}_{\max} - \sigma_{\min} \tag{2.5}$$

where the radial stress, as is usual in the theory of shells, is assumed to be zero. The values of $\sigma_{\max}$ and $\sigma_{\min}$ are taken from (2.4).

The stress values are calculated in the following range of parameters of the lens which is characteristic for the optical system of the focusing device [1]: current in the lenses $I_0 \sim 0.5$ mA, length of the lenses, L = (50–170) cm, end radius r_k = (7–25) cm, neck radius r_0 = (0.8–3) cm, constant of the parabola a = (0.05–1.5) cm^{-1}, thickness of the shell at the point $\theta = 0$ b_0 = (0.4–2) cm. In Fig. 3 we have presented the distribution, along the meridian s, of the stresses σ_{T_1}, σ_{T_2}, σ_{M_2} (curves 1, 2, 3, 4, respectively) for a lens with the parameters $I_0 = 0.5$ mA, $r_k = 9.5$ cm, $r_0 = 1.5$ cm, $h_0 = 1$ cm, $a = 0.404$ cm^{-1}. The solid curves are the result of numerical integration of Eqs. (2.1), the dashed lines are the result of analytical calculation according to (3.1), (3.9). The left-hand boundary of Fig. 3 corresponds to the neck, while the right-hand boundary corresponds to the flange of the lens.

A rigid fixing of the ends of the parabolic portion was assumed at the neck and at the flange. In all varients of the lens considered the total stress [according to expressions (2.4)] in the zones joining the lens with the neck and the flanges turned out to be the largest. At the neck they are the tensile stresses on the outer surface of the paraboloid, while at the flange they are the compressive stresses on the inner surface; at the same time the former are always considerably higher. When moving away from the fixing locations, the overall stresses rapidly decrease.

The effect of the fixing methods on the stress state was clarified. The following variants were considered: the flange rigidly fixed; displacements along the lens axis allowed; the lens has a preliminary compression (with respect to the supply of current); an absolutely free boundary (without the flange).

In the role of the boundary conditions at the neck we took $u^\circ = w^\circ = \vartheta_1 = 0$. A calculation showed that in the second case the stress state at the neck considerably worsens in comparison with the first. The stresses in the fourth variant remained approximately the same as in the second variant. The worsening of the stress state in the second and fourth versions in comparison with the first is connected with the fact that in the case of the flange being freed the total longitudinal force of the magnetic pressure $10^4 I_0^2 \ln r_k r_0^{-1}$ (kgf) is balanced by the forces T_1 and Q_1 only at the neck. (The radial constituent of the force is always self-balanced.) Hence, it follows that the first variant is more preferable. The preliminary compression of the lens (case three) weakens the stress state at the neck. But here the stability of the lens can worsen.

Below we present the results of calculations of the maximum equivalent stress σ_e^* (falling on the neck region), dependent on the geometrical parameters a, h_0, r_0. In Fig. 4 we have shown the dependence of σ_e^* on h_0 for $I_0 = 0.5$ mA, $r_k = 7.5$ cm, $r_0 = 1.5$ cm and various values of the parabola constant a (curve 1 corresponds to $a = 0.2$ cm^{-1}, 2) $a = 0.4$ cm^{-1}, 3) $a = 0.6$ cm^{-1}, 4) $a = 0.8$ cm^{-1}, 5) $a = 1$ cm^{-1}, 6) $a = 1.2$ cm^{-1}, 7) $a = 1.4$ cm^{-1}).

As follows from Fig. 4 [see also expressions (3.1)], the stresses in the case of a constant current and radius r_k decrease as the parameter a decreases and h_0 increases. This is connected basically with the variation of the normal thickness of the shell in the zone of contact with the neck, given by the expression $h = h_0(1 + 4a^2 r_0^2)^{-1/2}$. In accordance with this expression we should expect an increase of the stresses as the neck radius r_0 increases, since then h decreases. But as r_0 increases, the magnetic pressure ($\sim r_0^{-2}$) falls, and also the longitudinal constituent of the magnetic pressure decreases; as a result the stresses at the neck decrease as its radius increases. The dependence of the maximum stresses on the neck radius is presented in Fig. 5 for $I_0 = 0.5$ mA, $r_k = 9.5$ cm, $h_0 = 1$ cm, $a = 0.842$ cm^{-1}.

The transition to a parabolic lens with an axial hole in the neck leads to a decrease in the maximum stresses in the zone of contact between the paraboloids and the neck. For example, for a lens with the parameter $a = 0.1236$ cm^{-1}, the thickness of the paraboloid along the axis $h_0 = 1$ cm, and $r_0 = 3$ cm, the presence of a hole of radius 2.17 cm in the neck, under the condition that the neck and the paraboloid are equally stressed at the joint, reduces σ_e^* from 8 to 5.9 kgf/mm^2 (the numerical data are given in more detail in [5]).

3. For an approximate analytical solution of the problem we divide the stress state, following for example, [3], into a momentless state and a boundary effect. Putting $M_1 = M_2 = Q_1 = 0$ in (1.1) and taking into account (1.2), from the first two equations of the system of equations (1.1) we obtain

$$\sigma_{T_1}^\circ = 10^4 I_0^2 a (\pi h_0)^{-1} [1 + (4a^2r^2)^{-1}] (\ln r_k r^{-1} - 10^{-4} I_0^{-2} F)$$
$$\sigma_{T_2}^\circ = -10^4 I_0^2 a (\pi h_0)^{-1} [1 + (\ln r_k r^{-1} + 10^{-4} I_0^{-2} F)(4a^2r^2)^{-1}] \tag{3.1}$$

where $\sigma_{T_1}{}^\circ = T_1/h$, $\sigma_{T_2}{}^\circ = T_2/h$, and F is the constant of integration having the meaning of a force acting on the shell along the axis from the side of the flange and the neck. It is determined from the boundary conditions for T_1 or for the displacement u, when integrating the fourth and the fifth equations of system (1.1), with (3.1) taken into account. In the case where the shell is fixed at both ends, we must take the solution for the boundary conditions $u = 0$ at both boundaries in the role of the momentless solution. The constant of integration is determined as follows:

$$10^{-4} I_0^{-2} F = [{}^1/_2 (1 + 2\mu)(a^2 r_k^2 - a^2 r_0^2 + \ln r_k r_0^{-1}) + {}^1/_2 (1 + \mu) \ln 2 r_k r_0^{-1} + (1 + \mu - 8a^4 r_0^4)(8a^2 r_0^2)^{-1} \ln r_k r_0^{-1}] [a^2 r_k^2 - a^2 r_0^2 + (1 + \mu) \ln r_k r_0^{-1} + (1 + \mu)(8a^2 r_0^2)^{-1} - (1 + \mu)(8a^2 r_k^2)^{-1}]^{-1} \tag{3.2}$$

The equation of the boundary effect

$$d^4 w / d\theta^4 + k^4 w = 0, \quad k = [3(1 - \mu^2)]^{1/4} R_1 (R_2 h)^{-1/2} \tag{3.3}$$

is obtained from the homogeneous system (1.1) ($p_n = 0$), when neglecting the terms of order $(h/R)^{1/2}$.

The damped part of the solution of this equation has the form

$$w = e^{-k\theta} (c_1 \sin k\theta + c_2 \cos k\theta) \tag{3.4}$$

Here the moments and the shear force are connected with the displacement w by the relations

$$M_1 = \frac{D}{R_1^2} \frac{d^2 w}{d\theta^2}, \quad M_2 = \mu M_1, \quad Q_1 = \frac{1}{R_1} \frac{dM_1}{d\theta} \tag{3.5}$$

In the case of clamping the maximum bending stresses are

$$\sigma_{M_1} = \sqrt{3/(1 - \mu^2)}\, (\overset{\circ}{\sigma}_{T_2} - \mu \overset{\circ}{\sigma}_{T_1}) \tag{3.6}$$

Here $\sigma_{T_1}{}^\circ$ and $\sigma_{T_2}{}^\circ$ are taken from the momentless solution (3.1) for the corresponding boundary of the paraboloidal part of the lens ($r = r_0$ at the neck of the lens or $r = r_k$ at the flange). Since in this analysis the resultant force, acting on the shell from the side of the fixing, can have only a radial constituent (the longitudinal constituent is taken into account in the momentless solution), we have

$$Q_1 \cos \theta + T_1^k \sin \theta = 0 \tag{3.7}$$

and in the region of the boundaries we must take into account the force T_1^k which is supplementary to the momentless force T_1. The corresponding correction to the momentless stress $\sigma_{T_1}{}^\circ$ directly on the boundary equals

$$\sigma_{T_1}{}^k = {}^1/_3 (2ah_0)^{1/2} [3(1 - \mu^2)]^{1/4} \sigma_{M_1} \cos^2 \theta / \sin \theta \tag{3.8}$$

where σ_{M_1} and the angle θ are taken on the boundaries of the shell.

The total stresses on the boundaries of the shell equal

$$\sigma_1 = \pm \sigma_{M_1} + \sigma_{T_1}{}^\circ + \sigma_{T_1}{}^k, \quad \sigma_2 = \mu \sigma_1 \tag{3.9}$$

where the upper sign corresponds to the outer surface of the shell, while the lower corresponds to the inner surface.

From the analysis of the limits of applicability of the approximate solution [3], for the parabolic shells being considered we can obtain the criterion

$$1/\cos^3 \theta > [3(1 - \mu^2)]^{-1/4} [h_0 (2ar^2)^{-1}]^{1/2} \tag{3.10}$$

For shells which satisfy criterion (3.10), an analytical solution should be completely applicable and reliable. This is confirmed by comparison with the results of numerical solution of the total moment system of equations (1.1) (see, for example, Fig. 3).

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